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**LETTER TO THE EDITOR**

# Gauge fields associated with minimal surfaces and their Hopf invariants

J C Martinez
Math-Science Centre, Ngee Ann Polytechnic, Singapore 599589, Singapore

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**Abstract.** We study gauge fields described by an $(n-1)$-form $\alpha$ in a $(2n-1)$-dimensional Riemannian manifold satisfying the equations $\mathrm{d} * \alpha = 0$, $L_X\alpha = \mathrm{d}h$. We show that minimal surfaces may be associated with them that have a unique Hopf invariant. Some of these surfaces are stable but the issue is not settled completely.

Consider a divergence-free $(n-1)$-form $\alpha$ living in a $(2n-1)$-dimensional Riemannian manifold $M$ of fixed orientation. Hereafter, we will refer to $\alpha$ as a gauge field. Wherever necessary a metric will be assumed. Suppose now that $\alpha$ satisfies the equations

$$\mathrm{d} * \alpha = 0 \tag{1}$$

$$L_X\alpha = \mathrm{d}h \tag{2}$$

where $h$ is an $(n-2)$-form, $*\alpha$ is the Hodge dual of $\alpha$, $X$ is a divergence-free vector field and $L_X$ is the Lie derivative with respect to $X$. The conventions of Eguchi, Gilkey and Hanson [1] will be used below.

Examples of $\alpha$ occur often. For instance, Berry's adiabatic gauge field for a spin-$\frac{1}{2}$ particle in a constant magnetic field, namely $A = \frac{1}{2}(1+\cos)\,\mathrm{d}\phi$ in $\theta - \phi$ space satisfies (1) and (2) for $X = \partial/\partial\phi$. As is well known, $A$ is a connection of the Hopf bundle of $S^3$ over $S^2$ [2]. In Euclidean three-space the gauge field $A = \cos z\,\mathrm{d}x - \sin z\,\mathrm{d}y$ satisfies the above equations for $\alpha$ with $X = y(\partial/\partial x) - x(\partial/\partial y) + \partial/\partial z$. This describes a helicoidal gauge force-free field of infinite extent. An inviscid incompressible fluid can also be described within the same framework [3].

In this letter we wish to describe the properties of fields satisfying equations (1) and (2). In particular we will consider fields embedded in $S^3$ (Berry's example being a case in point).

First of all, the vector fields $X_i$ form a Lie algebra with respect to the Lie brackets [, ] because

$$L_{[X_i,X_j]}\alpha = (L_{X_i}L_{X_j} - L_{X_j}L_{X_i})\alpha = \mathrm{d}(L_{X_i} - L_{X_j})h \tag{3}$$

and $[X_i, X_j]$ is divergence-free if $X_i$ and $X_j$ are. The vectors $X_i$ constitute an involute vector space in this sense. Thus several symmetries may be simultaneously considered.

We introduce next the energy $E_B$ and the Hopf invariant $S_{AB}$

$$\begin{aligned} E_B &= \int_M *B \wedge B \equiv (B, B) \\ S_{AB} &= \int_M \alpha\,\mathrm{d}\alpha \end{aligned} \tag{4}$$

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where $B \equiv \mathrm{d}\alpha$ is an $n$-form. We will refer to $B$ as the magnetic field. $S_{AB}$ is a topological invariant which measures the helicity of fields (see below). Under transformations generated by $X$, $S_{AB}$ is an invariant, for by equations (1) and (2)

$$L_X S_{AB} = 0. \tag{5}$$

Furthermore, if $X$ is a Killing vector field, then it is known that $L_X * = * L_X$ so we have

$$L_X E_B = 2 \int *B \wedge L_X B = 0. \tag{6}$$

by equation (2), i.e. $E_B$ is invariant under flows arising from $X$ (isometries). Since Killing vector fields are divergence-free we conclude that $S_{AB}$ and $E_B$ are invariant under isometries.

Equation (2) translates to $L_X B = L_X \,\mathrm{d}\alpha = 0$. In as much as $B$ is an $n$-form, it describes an $n$-surface $S$ which we define here as the surface for which $B$ is proportional to the surface area $n$-form of $S$, i.e. $S$ is homologically trivial. Such surfaces may be termed magnetic surfaces and have been discussed in the literature [4].

We show below that $S$ corresponds to a minimal surface. Let us first show that $X$ need not be normal to $S$. Let $E_1, E_2, \ldots, E_n$ be a local orthonormal frame in $S$. From a well known formula [5]

$$(L_X B)(E_1, E_2, \ldots, E_n) = XB(E_1, E_2, \ldots, E_n) - \sum_{i=1}^{n} B(E_1, \ldots, [X, E_i], \ldots, E_n). \tag{7}$$

By virtue of our discussion above, the left-hand side of equation (7) vanishes. Since $B(E_1, E_2, \ldots, E_n)$ is the volume form and $X$ is volume-preserving, the first term on the right-hand side vanishes. It follows that the second term vanishes as well. On the other hand, we may write for the second term on the right

$$-\sum_i \sum_j B(E_1, \ldots, g([X, E_i], E_j)E_j, \ldots, E_n) = -B(E_1, \ldots, E_n) \sum_i g([X, E_i], E_i) = 0 \tag{8}$$

($g(\,,\,)$ denotes the metric). If the magnetic field $B(E_1, E_2, \ldots, E_n)$ does not vanish then $X$ is normal to $S$ and it belongs to the normal subspace of $S$. However, it may happen that the field vanishes and in this case $X$ need not commute with $E_i$. We shall examine this possibility later.

Now the mean curvature along a normal $N$ to $S$, denoted by $k(N)$, is

$$\begin{aligned}
k(N) &\equiv \frac{1}{n} \sum_i (\mathrm{D}_{E_i} E_i, N) = -\frac{1}{n} \sum_i (\mathrm{D}_{E_i} N, E_i) \\
&= -\frac{1}{n} \sum_i (\mathrm{D}_N E_i + [E_i, N], E_i) \\
&= -\frac{1}{n} \sum_i \frac{1}{2} \mathrm{D}_N (E_i, E_i) + \frac{1}{n} \sum_i g([N, E_i], E_i) \\
&= \frac{1}{n} \sum_i g([N, E_i], E_i).
\end{aligned} \tag{9}$$

Here $\mathrm{D}_V$ is the covariant derivative with respect to the vector field $V$. In the second line above we had used the fact that $\mathrm{D}_X Y - \mathrm{D}_Y X = [X, Y]$ [6]. Since $[N, E_i]$ vanishes, $k(N) = 0$ and so we have verified that $S$ is indeed a minimal surface.

Physically $L_X B = 0$ means the invariance of $B$ under flows generated by $X$. As the above calculation shows, the fields satisfying equations (1) and (2) issue normally through a minimal surface. The magnetic surfaces are analogues of the equipotential surfaces in electrodynamics or fluid mechanics, where these are important in their own right.

Because of our interpretation of $B$ as the volume form of $S$, the condition that $S$ be a minimal surface is equivalent, by equation (4), to the condition that energy is minimum. This is analogous to surface energy in soap bubbles. We can go further by showing that energy is a minimum when

$$\mathrm{d}\alpha = \lambda * \alpha \tag{10}$$

where $\lambda$ is a constant. (If we accept equation (1), then $\lambda$ can only be a constant and in this case it is clear that $E_B = \lambda S_{AB}$.)

To prove equation (10), consider the quantity $E_B = \lambda S_{AB} = \int \alpha \, \mathrm{d}\{(-)^n * \mathrm{d}\alpha - \lambda\alpha\}$. On varying with respect to $\alpha$, the above result readily emerges. To verify that we have indeed minimum field energy, define $I \equiv (\alpha, \alpha)$. $S_{AB}$ satisfies the Schwarz inequality $S_{AB}^2 \leqslant I \cdot E_B$ with equality holding precisely when equation (10) is true. Since $S_{AB}$ is a topological invariant, it follows that equation (10) is indeed the condition for minimum energy. (One may add to the right-hand side of equation (10) an $n$-form $*\mathrm{d}\beta$ but this contributes a total divergence to the field energy and may be ignored.) In the minimal-energy case, one can show directly from equation (10) that

$$\Delta\alpha = \lambda^2\alpha \tag{11}$$

where $\Delta = \delta\mathrm{d} + \mathrm{d}\delta$ is the Laplacian, and $\delta = (-)^{pn+n+1} * \mathrm{d}*$ the co-derivative acting on a $p$-form in $n$-space.

Finally, we give some remarks about the Hopf invariant. The Hopf invariant gives the linking number of two disjoint manifolds and is a genuine topological invariant [7]. In the minimal energy case, if we normalize $\alpha$ such that $(\alpha, \alpha) = 1$, then $\lambda$ is an integer, namely the linking number. The Hopf invariant classifies manifolds into equivalence classes. We can further appreciate the relevance of $S_{AB}$ as follows. In $R^3$ let $C$ be a closed curve on a surface $S$ and let the vector field $A$ be everywhere normal to $S$. Then the loop integral $\int_C A \, \mathrm{d}l$ vanishes. By Stokes' theorem this integral is also equal to $\int_{S'} \nabla \times A \cdot \mathrm{d}\sigma$ where $C$ bounds the surface $S' \subset S$. But this surface integral does not vanish in general unless $A \cdot \nabla \times A = 0$ everywhere. This translates, in simply connected three-space, to the impossibility of constructing (even locally) surfaces that are normal to $A$ unless the condition $A \cdot \nabla \times A = 0$ holds. We shall show below that this is equivalent to the vanishing of the Hopf invariant. Thus the nonvanishing of the Hopf invariant signals a nontrivial geometric structure. In the minimal-energy case, equation (10) indicates that a nontrivial manifold structure is involved.

It is known that in Euclidean space there are no compact minimal surfaces. In fact it can be shown that the helicoid is the only minimal surface in Euclidean space satisfying equations (1) and (2). Our discussion below will be limited to compact fields.

From this point on, we focus attention on gauge fields in $S^3$. First we show by a different method that the association with minimal surfaces holds for such gauge fields. Let $X$ be a Killing field and let $f = \frac{1}{2}\langle X, X\rangle$ be half of its length squared. Guided by the work above we compute the Laplacian of $f$, defined here alternatively as a contraction

$$\Delta f = \sum_i \langle \mathrm{D}_{E_i} \operatorname{grad} f, E_i\rangle. \tag{12}$$

From the definition $\langle \operatorname{grad} f, X\rangle = Xf$, valid for any vector field $X$, we obtain $\operatorname{grad} f = -\mathrm{D}_X X$. Now we invoke the definition of the Riemannian curvature tensor $R$ for a set of vectors, $V$, $W$ and $X$ [6]:

$$\langle \mathrm{D}_V \mathrm{D}_X X, W\rangle = \langle \mathrm{D}_{[V,X]} X - R_{VX} X + \mathrm{D}_X \mathrm{D}_V X, W\rangle. \tag{13}$$

Because $X$ is a Killing vector, i.e. it satisfies $\langle \mathrm{D}_V X, X\rangle + \langle \mathrm{D}_X X, V\rangle = 0$, the last two equations yield

$$\Delta f = -\operatorname{Ric}(X, X) + \operatorname{trace}(\mathrm{D}X, \mathrm{D}X), \tag{14}$$

where $\operatorname{Ric}(X, X) = \sum_{E_i} \langle R_{XE_i} X, E_i\rangle$ is the Ricci tensor. Equation (14) holds for any Killing vector $X$. On the sphere, however, $X = x^i \partial_j - x^j \partial_i$ so that $f$ has the simple form $\frac{1}{2}(x^i)^2 + \frac{1}{2}(x^j)^2$. Also $\mathrm{D}X$ is zero or normal to the sphere so $\operatorname{tr}(\mathrm{D}X, \mathrm{D}X)$ vanishes. Because the sphere $S^3$ has constant sectional curvature, we have

$$\operatorname{Ric}(X, X) = 2\langle X, X\rangle / r^2 \tag{15}$$

where $r = 1$ is the radius of $S^3$. With the $f$ given above, the left-hand side of equation (14) may be cast as $\langle \mathrm{D}x^i, \mathrm{D}x^i\rangle + \langle x^i, \Delta x^i\rangle$ and finally we obtain

$$\Delta x^i = -2x^i \tag{16}$$

as the equation for the components occurring in the Killing vector $X$ which generates isometry transformations on the fields. Alternatively, the $x^i$ describe the coordinates of the fields embedded in $S^3$ upon which $X$ operates. (In Euclidean space we have $r \to \infty$ so equation (16) is replaced by $\Delta x^i = 0$. One can verify that the helicoid mentioned in the introduction satisfies it.)

It is well known that equation (16) is the equation of a minimal $n = 2$ surface embedded in $S^3$ [8]. Thus we have shown that the gauge fields obeying equations (1) and (2) and embedded in $S^3$ correspond to minimal surfaces.

To illustrate this, let us consider Berry's example. It is really a Dirac monopole in parameter space. The Dirac vector potential is just the connection of $S^3$, the principal $U(1)$ bundle over $S^2$ [9]. If we think of $S^3$ as a subset of $C^2 = C \times C$, the Killing field is generated by the rotation of one of the complex factors and is tangent to two circles.

We employ real coordinates $y^1, y^2, y^3, y^4$ on $C^2$ with $(y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 = 1$ for $S^3$. The corresponding Dirac (connection) 1-form is

$$A = \frac{1}{\pi}(y^1\,\mathrm{d}y^2 + y^3\,\mathrm{d}y^4). \tag{17}$$

In $S^3$, the volume form is proportional to

$$\sigma = y^1\,\mathrm{d}y^2 \wedge \mathrm{d}y^3 \wedge \mathrm{d}y^4 - y^2\,\mathrm{d}y^1 \wedge \mathrm{d}y^3 \wedge \mathrm{d}y^4 + y^3\,\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^4 - y^4\,\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3$$

so we find $\mathrm{d}A = *A$, and of course $\mathrm{d} * A = 0$. Moreover

$$\Delta A = A \tag{18}$$

and the Hopf invariant is

$$\begin{aligned} \int_{S^3} A\,\mathrm{d}A &= \frac{2}{\pi^2}\int_{S^3} y^1\,\mathrm{d}y^2\,\mathrm{d}y^3\,\mathrm{d}y^4 \\ &= \int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} \sin^4\alpha \sin^3\phi \cos^2\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\alpha = 1 \end{aligned} \tag{19}$$

where the spherical coordinates $x_1 = \sin\alpha\sin\phi\cos\theta$, $x_2 = \sin\alpha\sin\phi\sin\theta$, $x_3 = \sin\alpha\cos\phi$, $x_4 = \cos\alpha$ were introduced. Thus we have verified equations (4) and (11) together with the linking number interpretation of $\lambda$.

To obtain the corresponding magnetic surfaces, we employ the Hopf map $\pi : S^3 \to S^2$ [9]. Parametrizing $S^3$ by $z_1 = \cos\frac{1}{2}\theta \mathrm{e}^{\mathrm{i}\Psi_1}$, $z_2 = \sin\frac{1}{2}\theta \mathrm{e}^{\mathrm{i}\Psi_2}$, we have

$$\begin{aligned} \pi(z_1, z_2) &= (z_1^* z_2 + z_2^* z_1, -\mathrm{i}z_1^* z^2 + \mathrm{i}z_2^* z_1, |z_1|^2 - |z_2|^2) \\ &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \end{aligned} \tag{20}$$

where $\phi = \Psi_2 - \Psi_1$. If we set $(x_1, x_2, x_3, x_4) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta, 0)$ as the coordinates of the magnetic surface in $S^3$, we find that

$$\Delta x_i = -2x_i \qquad i = 1, 2, 3, 4. \tag{21}$$

This verifies equation (16) and, of course, the surface is a patch on $S^2$.

In 1970 Lawson [10] showed that there are minimal embeddings into $S^3$ of surfaces of arbitrary genus. It would be interesting to relate these to physically realizable systems.

We show next that gauge fields in $S^3$ that have the same Hopf invariant are equivalent up to a gauge transformation. A Riemannian metric will be understood to have been introduced. Thus, suppose that two gauge fields $A$ and $B$ exist such that

$$\int_{S^3} A\,\mathrm{d}A = \int_{S^3} B\,\mathrm{d}B = n. \tag{22}$$

It is easy to verify from this and the fact that $S^3$ is closed that

$$\int_{S^3} (A + B)\,\mathrm{d}(A - B) = 0. \tag{23}$$

We now show that for arbitrary fields $\omega$ and $\bar{\omega}$ in $S^3$ satisfying the condition

$$\int \omega\,\mathrm{d}\bar{\omega} = 0 \tag{24}$$

one of them must vanish or be exact.

Any gauge field $\omega$ in a compact and closed manifold can be written as a sum of its harmonic part $\omega^0$, its longitudinal part $\mathrm{d}\alpha$ and its transverse part $\delta\beta$. By virtue of equation (2), the first two vanish so we have simply $\omega = \delta\beta$, i.e. $\omega$ is transverse. Now we expand $\omega$ in terms of a set of a normalized basis of transverse eigenforms $\phi_n$ of the Laplacian $\Delta : \Delta\varphi_n = \lambda_n^2\varphi_n$, $\delta\varphi_n = 0$, $(\varphi_n, \varphi_m) = \delta_{nm}$. The eigenvalues are nonvanishing. Thus,

$$\omega = \sum_{n=1}^{\infty} c_n \varphi_n \tag{25}$$

where the $c_n$ are expansion coefficients. We can also define another normalized basis of transverse eigenforms $\phi_n$ for $\bar{\omega}$:

$$\phi_n = -\frac{1}{\lambda_n} * \mathrm{d}\varphi_n. \tag{26}$$

This is possible because the Hodge dual provides a natural isomorphism between the space of 1-forms and the corresponding space of 2-forms in compact 3-space. Observe also that $*\mathrm{d}$ is linear and invertible. We can verify that $\Delta\phi_n = \lambda_n^2\phi_n, (\phi_n, \phi_m) = \delta_{nm}, \delta\phi_n = 0$. Hence we may write a parallel expansion for $\bar{\omega}$:

$$\bar{\omega} = \sum_{n=1}^{\infty} b_n \phi_n. \tag{27}$$

We now evaluate equation (24):

$$0 = \int_{S^3} \omega\,\mathrm{d}\bar{\omega} = \sum_{n,m} c_n b_m \int_{S^3} \varphi_n\,\mathrm{d}\phi_m = \sum_n c_n b_n \lambda_n \tag{28}$$

where we had used the various properties of the eigenforms defined above. Since the eigenvalues $\lambda_n$ are nonvanishing and can be arranged in ascending order, this result holds if and only if all the expansion coefficients $c_n$ or all the expansion coefficients $b_n$ vanish. Therefore equation (23) holds if and only if

$$A = \pm B. \tag{29}$$

Because the Hopf invariant is gauge invariant we may relax this condition to

$$A = \pm B + \mathrm{d}f \tag{30}$$

where $f$ is a smooth 0-form. This shows that two gauge fields satisfying equation (22) are essentially gauge equivalent.

Two remarks are in order. First, consider equation (8) once again in $S^3$. If the magnetic field vanishes, then $\sum g([X, E_i], E_i)$ need not vanish. Thus $X$ need not be normal to the magnetic surface. It is known that a compact simply connected three-manifold has rank one, that is, two commuting vectors must necessarily be linearly dependent somewhere in the manifold (Lima's theorem). A sufficient condition that a three-manifold be simply connected is that $A\,\mathrm{d}A$ vanishes. Thus a three-dimensional compact space in which $A\,\mathrm{d}A = 0$ has zero magnetic field everywhere. It is not difficult to construct a field in $S^3$ with zero linking number, namely $A = \frac{1}{r}\mathbf{1}\hat{\phi}$, where $\mathbf{1}$ is the $2 \times 2$ identity matrix. The magnetic field $B = \nabla \times A$ vanishes for this case. Since we know now that two fields with zero linking number are gauge equivalent, it follows that gauge fields with zero linking number in $S^3$ must necessarily have vanishing magnetic field. Secondly, the argument above was cast for three-dimensional space. It can be extended to any odd-dimensional compact space. The only modification is in the definition (26), which should read instead,

$$\phi_n = (-)^{2p-1-p-1}\frac{1}{\lambda_n} * \mathrm{d}\varphi_n \tag{31}$$

for $p$-eigenforms in a $(2p-1)$-dimensional compact manifold.

A final characteristic of the surfaces we have been considering will be discussed. It is widely known that most minimal surfaces are really unstable, that is, their second variation is negative. We also observed in equation (6) that the energy density $B \wedge *B$ is invariant under $X$, so we need consider only normal variations of the surface area $\int \mathrm{d}\sigma$. It is known that $\delta \int \mathrm{d}\sigma = -2\int HV\,\mathrm{d}\sigma$, where $V$ is the normal variation of the surface $\boldsymbol{x}(u, v) \to \boldsymbol{x}(u, v) + tV(u, v)\hat{\boldsymbol{n}}$, $t \in (-\varepsilon, \varepsilon)$ [8]. Because $H$ vanishes for minimal surfaces, we see that the first variation of the surface area vanishes and only the second variation of the area is required (the energy density remaining invariant):

$$\delta^2 E_B = -2(B \wedge *B\delta H). \tag{32}$$

For compact and closed two-manifolds in $S^3$ it can be shown that [11]

$$\delta H = \tfrac{1}{2}(\Delta V + 2KV) \tag{33}$$

where $K = 1$ is the constant sectional curvature of $S^3$ and $\Delta$ the Laplacian on the surface. Limiting ourselves to normal variations proportional to the coordinates of the surface, by virtue of equation (16), we find $\delta H = 0$ and so for gauge fields in $S^3$

$$\delta^2 E_B = 0. \tag{34}$$

For the case of the helicoidal field in $R^3$, which is non-compact, the corresponding normal variation of $H$ is [8]

$$\delta H = \tfrac{1}{2}(\Delta V + 4H^2 V - 2KV) \tag{35}$$

where $H = 0$ for a helicoid and $K (< 0)$ is its Gaussian curvature. Since minimal surfaces in $R^3$ satisfy $\Delta x = 0$, we find then that

$$\delta^2 E_B < 0. \tag{36}$$

Thus, we have established the stability of the helicoid. But gauge fields in $S^3$ appear to require further study. It would be necessary to consider more general surface and field variations.

In conclusion, we have shown that minimal surfaces may be associated with fields characterized by equations (1) and (2) and that in $S^3$ such fields have a unique Hopf invariant. The issue of stability of the surfaces is not settled: in $R^3$ it seems that the surfaces are stable, whereas in $S^3$ further study of higher variations is required.

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